

A new simple class of superpotentials in SUSY Quantum Mechanics

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In this work we introduce the class of quantum mechanics superpotentials $W(x) = g\varepsilon(x)x^{2n}$ and study in details the cases $n = 0$ and 1 . The $n = 0$ superpotential is shown to lead to the known problem of two supersymmetrically related Dirac delta potentials (well and barrier). The $n = 1$ case result in the potentials $V_{\pm}(x) = g^2x^4 \pm 2g|x|$. For V_- we present the exact ground state solution and study the excited states by a variational technic. Starting from the ground state of V_- and using logarithmic perturbation theory we study the ground states of V_+ and also of $V(x) = g^2x^4$ and compare the result got by this new way with other results for this last potential in the literature.

I. INTRODUCTION

Supersymmetric quantum mechanics (SUSY QM) was first introduced by E. Witten [1] [2], as a simplified model (a $0 + 1$ dimensional field theory) to study the possibility of SUSY breaking. Soon it became a research branch in itself, a way of getting new solutions to problems in quantum mechanics [3] [4] [5] [6]. Of particular interest, to our work below, we must cite the many papers in the literature [7] [8] [9] [10] [11] [12] [13] [14] devoted to the development of technics for treating the anharmonic oscillator $V(x) = \omega^2x^2 + g^2x^4$, and other related potentials, which in general do not have exact solutions.

In this work we present a new simple class of superpotentials in SUSY QM, in the form $W(x) = g\varepsilon(x)x^{2n}$ with $n = 0, 1, 2, \dots$. The first example of this class, i.e., the case $n = 0$,

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was studied long ago in [15] and revisited in [16] and [17]. One of our results is an analytic solution for the ground state wave function of the potential $V(x) = g^2x^4 - 2g|x|$, an amazing result, considering that analytic solutions do not exist for anharmonic oscillators. Another result is a new perturbative solution for the ground state of the potential $V(x) = gx^4$, starting from the solution for the potential $V(x) = g^2x^4 - 2g|x|$. Excited states of the potentials $V_{\pm}(x) = g^2x^4 \pm 2g|x|$ are also studied by a variational approximation.

The paper is organized as follows. In Sec. II, we make a brief introduction to the well known case of superpotentials, which are monomials of odd powers of x , as well as to the SUSY breaking ones, which are monomials of even powers of x . More details of these solutions can be found in [3] and [18]. In Sec. III, we study solutions related to the class of simple superpotentials of the form $W(x) = g\varepsilon(x)x^{2n}$ ($n = 0, 1, 2, \dots$), where $\varepsilon(x)$ is the sign function. The simple analytical solution for the ground state of the corresponding SUSY system is shown, the already known case $n = 0$ is revised and the case $n = 1$ is studied in more details. The first one is the illustrative example of the Dirac delta well and barrier potentials, which are shown to be SUSY partner potentials associated with the superpotential $W(x) = g\varepsilon(x)$. The second one, $W(x) = g\varepsilon(x)x^2$, allows us to find an analytical solution for the ground state of the potential $V(x) = g^2x^4 - 2g|x|$. In Sec. III A and Sec. III B, we study the excited states of the potentials $V_{\pm}(x) = g^2x^4 \pm 2g|x|$ which are derived from $W(x) = g\varepsilon(x)x^2$. After discussing that exact solutions for the excited states cannot be obtained, we apply a variational method (Sec. III A) to find approximate solutions for the energy levels and the wave functions. In Sec. III B, a new perturbative approach to the ground state of the potentials $V(x) = gx^4$ and $V(x) = g^2x^4 + 2g|x|$ is presented. Finally, a discussion of the results is presented in the Conclusions.

II. OUR NOTATION AND DEFINITIONS ON SUSY QM

Let us briefly summarize some main concepts in SUSY QM. For simplicity, we will work in a system of units with the Planck's constant set as $\hbar = 1$ and the particle mass set as $m = 1/2$ (that is $2m = 1$). We start by defining the operators A^\dagger and A :

$$A^\dagger = W(x) - ip \quad \text{and} \quad A = W(x) + ip, \quad (1)$$

where $W(x)$ is a given function of x and $p = -id/dx$ is the momentum operator. From these operators we can construct two hamiltonians:

$$H_- = A^\dagger A \quad \text{and} \quad H_+ = AA^\dagger, \quad (2)$$

which in terms of p and $W(x)$ result in: $H_\pm = p^2 + V_\pm$. The potentials V_\pm are given by the equations ($W' \equiv dW/dx$):

$$V_\pm(x) = W(x)^2 \pm W'(x), \quad (3)$$

which are Riccati's equations.

These equations can be understood in two ways. One way is: given $W(x)$, we can define the hamiltonians H_\pm with potentials $V_\pm(x)$. The other is: given the potential $V_-(x)$ (or $V_+(x)$), by solving one of the Riccati's equations, $W(x)$ can be found, the operators A and A^\dagger constructed and the partner potential $V_+(x)$ (or $V_-(x)$) can be found.

The ground state of a SUSY system is defined as the zero energy state of H_- (this is a choice; changing the function $W(x) \rightarrow -W(x)$ will change the roles of H_- and H_+). As $H_- = A^\dagger A$, its ground state wave function $\psi_0^-(x)$ can be got by imposing that it is annihilated by the operator A , that is:

$$A\psi_0^-(x) = \left(W(x) + \frac{d}{dx}\right)\psi_0^-(x) = 0.$$

The solution is given by:

$$\psi_0^-(x) = \mathcal{N} \exp \left\{ - \int^x W(y) dy \right\}. \quad (4)$$

This is a good, physically meaningful solution, provided that the function (4) is normalizable. Otherwise, a zero energy solution does not exist and SUSY is said to be broken. As it is easy to see, superpotentials obeying the rule of being positive ($W(x) > 0$) for $x > 0$ and negative ($W(x) < 0$) for $x < 0$ shall manifest SUSY.

Then, starting from $W(x)$, we have two partner hamiltonians, H_- and H_+ , one of them (H_- , in our choice) having a ground state ψ_0^- with energy $E_0^- = 0$ and a tower of other states: bound states with energies $E_n^- > 0$, $n = 1, 2, 3, \dots$ or scattering states with energies $E^- > 0$. The hamiltonian H_+ has bound energy levels E_{n-1}^+ , $n = 1, 2, 3, \dots$ with energies related to the energies of H_- by the relation: $E_{n-1}^+ = E_n^-$ or scattering energies $E^+ > 0$.

Moreover, the eigenfunctions of H_- and H_+ are related according to:

$$\psi_{n-1}^+ = (E_n^-)^{-1/2} A \psi_n^- \quad (5)$$

$$\psi_n^- = (E_{n-1}^+)^{-1/2} A^\dagger \psi_{n-1}^+ . \quad (6)$$

The simplest class of superpotentials manifesting supersymmetry are monomials of odd power in x , that is:

$$W(x) = gx^{2n+1} , \quad n = 0, 1, 2, \dots . \quad (7)$$

Using the Riccati equation (3), we have for the partner potentials:

$$V_\pm(x) = W(x)^2 \pm W'(x) = g^2 x^{4n+2} \pm g(2n+1)x^{2n} . \quad (8)$$

The ground (normalizable) state of $H_- = p^2 + V_-$, with energy $E_0^- = 0$ (see Eq. (4)) is given by:

$$\psi_0^-(x) = \mathcal{N} \exp \left\{ \frac{-gx^{2n+2}}{(2n+2)} \right\} . \quad (9)$$

The first example of a superpotential of the class (7) is $W(x) = gx$. In this case, the associated partner potentials are:

$$V_\pm(x) = g^2 x^2 \pm g , \quad (10)$$

which are simply the potentials of two harmonic oscillators of the same frequency, with a constant energy shift g added or subtracted. The ground state of H_- have $E_0^- = 0$. Its excited states and the states of H_+ are given by $E_n^- = E_{n-1}^+ = 2ng$, for $n = 1, 2, 3, \dots$

We will not pursue the study of this class of superpotentials because they are well known. We only mention that the next example of this class, $W(x) = gx^3$, corresponds to the potentials $V_\pm(x) = g^2 x^6 \pm 3gx^2$ and their ground state solution is given by (9) with $n = 1$.

On the other hand, the class of superpotentials that are monomials in even powers of x , does not give a normalizable zero energy solution to (4) and SUSY is broken. However, we can introduce the sign function $\varepsilon(x)$ and consider superpotentials of the form $W(x) = g\varepsilon(x)x^{2n}$. For this class of superpotentials, a normalizable ground state exists and SUSY is not broken. Thus, in the following we study this class of superpotentials, specially the $n = 0$ and $n = 1$ cases.

III. THE CLASS OF SUPERPOTENTIALS OF THE FORM $W(x) = g\varepsilon(x)x^{2n}$

The case $n = 0$ must be treated separately. So, let us consider the superpotential:

$$W(x) = g\varepsilon(x) , \quad (11)$$

where g is a positive constant. For this superpotential (11) the Riccati equations (3) give the following SUSY partner potentials:

$$V_{\pm}(x) = W(x)^2 \pm W'(x) = \pm 2g\delta(x) + g^2 , \quad (12)$$

where $\delta(x)$ is the Dirac delta function. V_- is a delta well, while V_+ is a delta barrier, with the energy of the ground state displaced by g^2 . The corresponding Schrödinger equations are:

$$-\psi^{\pm\prime\prime}(x) \pm 2g\delta(x)\psi^{\pm}(x) = (E^{\pm} - g^2)\psi^{\pm}(x) . \quad (13)$$

Their solutions are well known [15] [16] [17]. The well (V_-) has a single bound state with energy level $E_0^- = 0$, binding energy g^2 , and wave function given by:

$$\psi_0^-(x) = \sqrt{g}e^{-g|x|} . \quad (14)$$

All the other eigenstates are plane waves in continuous spectra of energies, the lowest one starting with $E = g^2$. Simple scattering solutions of the well V_- and the barrier V_+ can be written as:

$$\psi_I^{\pm}(x) = \mathcal{A}_{\pm}e^{ikx} + \mathcal{B}_{\pm}e^{-ikx} , \quad x \leq 0 \quad (15)$$

$$\psi_{II}^{\pm}(x) = \mathcal{C}_{\pm}e^{ikx} + \mathcal{D}_{\pm}e^{-ikx} , \quad x \geq 0 , \quad (16)$$

where $k = \sqrt{E^{\pm} - g^2}$ with $E^{\pm} > g^2$ and the respective constants are related according to the boundary conditions $\psi_{II}(0) = \psi_I(0)$ and $\psi'_{II}(0) = \psi'_I(0) \pm 2g\psi(0)$ required by the Dirac delta potential.

Summarizing: the hamiltonian H_- has one ground state with energy $E_0^- = 0$ and continuum of states with energies $E^- > g^2$ and H_+ has a continuum of states with $E^+ > g^2$.

To see the role of the supersymmetry in this system, let us consider a particle crossing the well (or hitting the barrier), coming from $x = -\infty$, such that we can choose $\mathcal{D}_{\pm} = 0$. With the appropriate boundary conditions through $x = 0$, we can determine \mathcal{B}_{\pm} and \mathcal{C}_{\pm} , getting

the scattered and the transmitted solutions as functions of the incident amplitudes \mathcal{A}_\pm . The results can be written as:

$$\psi_I^\pm(x) = \mathcal{A}_\pm \left\{ e^{ikx} + i \frac{\frac{(\mp g)}{k}}{\left(1 - i \frac{(\mp g)}{k}\right)} e^{-ikx} \right\}, \quad x \leq 0 \quad (17)$$

$$\psi_{II}^\pm(x) = \mathcal{A}_\pm \frac{1}{\left(1 - i \frac{(\mp g)}{k}\right)} e^{ikx}, \quad x \geq 0. \quad (18)$$

It is easy to verify that the solutions ψ^- and ψ^+ are related by the supersymmetry equations (5) and (6). For example, by applying the operator A to $\psi_I^-(x)$ we get:

$$\begin{aligned} A\psi_I^-(x) &\propto \left(g\varepsilon(x) + \frac{d}{dx} \right) \left\{ e^{ikx} + i \frac{\frac{g}{k}}{\left(1 - i \frac{g}{k}\right)} e^{-ikx} \right\} \\ &\propto \left\{ e^{ikx} + i \frac{\frac{-g}{k}}{\left(1 - i \frac{-g}{k}\right)} e^{-ikx} \right\} \propto \psi_I^+(x), \end{aligned}$$

explicitly showing the manifestation of the supersymmetry of the system.

Let us now consider the superpotential:

$$W(x) = g\varepsilon(x)x^2, \quad (19)$$

where here also, g is a positive constant. The two partner potentials are given by:

$$V_\pm(x) = W(x)^2 \pm W'(x) = g^2 x^4 \pm 2g|x|. \quad (20)$$

In these potentials a term $\delta V = \pm 2gx^2\delta(x)$ has been dropped. The reason is that for the wave functions involved in this problem its action is null. As the potentials $V_\pm(x) \rightarrow \infty$ for $x \rightarrow \pm\infty$, the spectra of $H_\pm = p^2 + V_\pm$ are discrete and their eigenfunctions are normalizable. If δV is treated as a perturbative correction to H_\pm , its action would be non null only if $\int_{-\infty}^{\infty} dx x^2 \delta(x) |\psi(x)|^2 \neq 0$. But this condition requires a wave function that near $x = 0$ behaves like $f(x)/x$ with $f(0) \neq 0$, which is non normalizable and is not in the spectra of H_\pm . On the other side, treated as part of H_\pm , the term δV could give non trivial boundary conditions for $d\psi/dx$ at $x = 0$. To study this possibility we must integrate the Schrödinger equation in the interval $x = (-\epsilon, \epsilon)$ for $\epsilon \rightarrow 0$. A non null effect of δV only comes if $\int_{-\epsilon}^{\epsilon} dx x^2 \delta(x) \psi(x) \neq 0$, which would require a $\psi(x)$ behaving like $f(x)/x^2$ with $f(0) \neq 0$ that is also, out of the spectra of H_\pm .

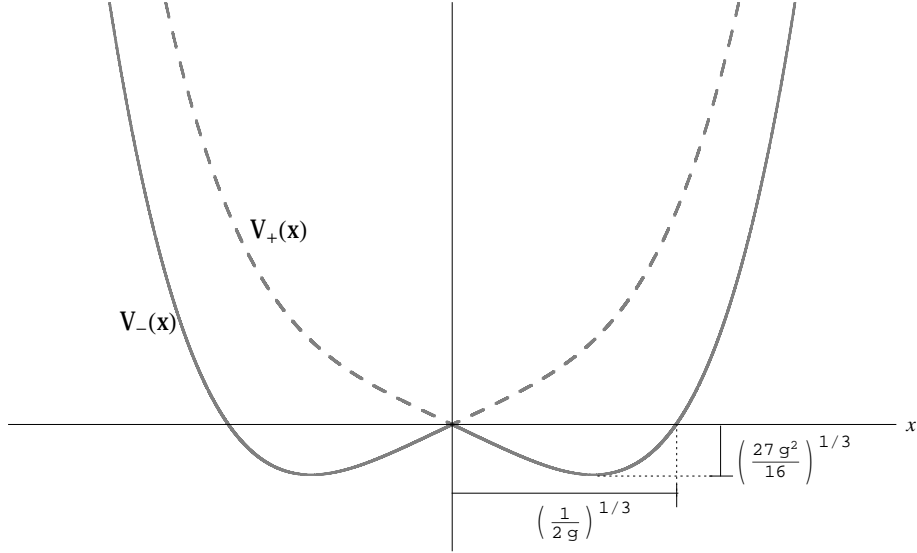


FIG. 1: Partner potentials $V_-(x)$ and $V_+(x)$ associated with the superpotential $W(x) = g\varepsilon(x)x^2$.

A representation of these potentials is given in Fig. 1. As can be seen, V_+ is a single well potential and V_- a double well potential symmetric in x . The corresponding Schrödinger equations read:

$$\left(-\frac{d^2}{dx^2} + g^2x^4 \pm 2g|x|\right) \psi^\pm(x) = E^\pm \psi^\pm(x). \quad (21)$$

The wave function for the ground state of the double well potential $V_-(x) = g^2x^4 - 2g|x|$, has energy $E_0^- = 0$ and is easily obtained from the equation:

$$0 = A\psi_0 = \left(g\varepsilon(x)x^2 + \frac{d}{dx}\right) \psi_0.$$

The result (already normalized) is given by:

$$\psi_0(x) = \left(\frac{3}{2}\right)^{1/3} \frac{g^{1/6}}{\Gamma(1/3)^{1/2}} e^{-g|x|^3/3}. \quad (22)$$

This is an interesting result. As it is well known, exact analytic solutions for the ground (or any excited) state of the potentials $V(x) = g^2x^4$ or $V(x) = \omega^2x^2 + g^2x^4$ cannot be obtained. So this exact solution for the potential V_- is somewhat surprising. Another characteristic of this solution, is that it represents a single lump centered at $x = 0$ (which is a local maximum of V_-) and it is not in the form, as naively expected, of two lumps centered at the two symmetric minima, $x = \pm(1/2g)^{1/3}$, of V_- notwithstanding the fact that, in one dimension, any attractive well supports at least a bound state. This happens because the "volume" of each well is not big enough to support a bound state (this can be seen in a

WKB analysis of the potential, or even more simply, by the Heisenberg uncertainty principle. We should only observe that this well size $\Delta x(\Delta E)^{1/2}$ is independent of g .

Let us now look for the excited states solutions. Inspired by the analytic method to solve the one-dimensional simple harmonic oscillator and by the form of the solution (22), we try a solution of the form ¹:

$$\psi(x) = F(x)e^{-g|x|^3/3} . \quad (23)$$

Substituting (23) in the Schrödinger equation (21), it becomes:

$$F'' - 2g\varepsilon(x)x^2F'(x) + EF(x) = 0 . \quad (24)$$

For the simple harmonic oscillator, the same steps would lead us to the Hermite equation. In our case, we get the equation (24), which is, for a particular choice of parameters, the Triconfluent Heun equation[19].

We can go on, look for solutions for the equation (24) through a power series method. Assuming that $F(x)$ can be written as:

$$F(x) = \sum_{j=0}^{\infty} a_j x^j \quad (25)$$

and substituting this expression for $F(x)$ in the differential equation (24), we find:

$$\sum_{j=0}^{\infty} j(j-1)a_j x^{j-2} - 2g\varepsilon(x) \sum_{j=0}^{\infty} j a_j x^{j+1} + E \sum_{j=0}^{\infty} a_j x^j = 0 .$$

Renaming indices and rearranging terms, we have:

$$2a_2 + Ea_0 + \sum_{j=1}^{\infty} [(j+2)(j+1)a_{j+2} - 2g\varepsilon(x)(j-1)a_{j-1} + Ea_j] = 0 .$$

Then, given a_0 e a_1 , this equation is satisfied if the coefficients a_j , $j \geq 2$, are given by the three terms recursion relations:

$$a_2 = -\frac{E}{2}a_0 , \quad j = 2 \quad (26)$$

$$a_j = \frac{2g\varepsilon(x)(j-3)a_{j-3} - Ea_{j-2}}{j(j-1)} , \quad j \geq 3 . \quad (27)$$

¹ In the case of the simple harmonic oscillator, we suppose that the solutions are of the form $H(x)e^{-x^2/2}$ and, imposing that those solutions are square integrable, the functions $H(x)$ becomes restricted to be the Hermite polynomials $\mathcal{H}_n(x^2)$.

The corresponding recursion relation for the harmonic oscillator potential, is a simple two terms recursion relation. To get a normalizable solution, we choose the values of E so as to terminate the series in a polynomial. In this way we get the set of discretized values of the energy spectrum and the corresponding wave functions, that turn up to be the Hermite polynomials (see footnote).

In our case, the recurrence relation (27), is a three terms recurrence relation and there is no way of choosing a subset of values of E to terminate the series in polynomials, so as to have a normalizable solution. Then, no analytic solution can be found and in the next sections we pass to look for approximate solutions. In Sec. III A a variational approximation is studied and in Sec. III B a perturbative approximation, that will allow us also, to study solutions for the potential $V(x) = gx^4$.

A. Looking for Approximate Solutions by a Variational Method

Let us first apply a variational method. The trial function that we are going to use is:

$$\phi(x) = \sum_{j=1}^m \alpha_j f_j(x) , \quad (28)$$

where $j = 1, 2, \dots, m$. and the coefficients $\alpha_j \in \mathbb{C}$ are the variational parameters. The functions $f_j(x)$ are chosen to be:

$$f_j(x) = x^{j-1} e^{-g|x|^3/3} . \quad (29)$$

This trial function corresponds to the previously used in the power series method, with the additional restriction of being a finite polynomial of degree $m - 1$, instead of an infinite series in x .

For the harmonic oscillator, with a very similar choice of the trial function we would find exact solutions. In that case, the variational parameters would be, except for the normalization, the coefficients of the Hermite polynomials $\mathcal{H}_n(x^2)$.

Before proceeding, let us consider a convenient change of variables. As can easily be seen, by making the rescaling: $x \rightarrow g^{-1/3}x$ it is possible to factor out of the hamiltonians H_{\pm} , the constant $g^{2/3}$, that is:

$$H_{\pm} = g^{2/3} \left(-\frac{d^2}{dx^2} + x^4 \pm 2|x| \right) . \quad (30)$$

So, in the rest of this section, we will work with $g = 1$ and after finding the energy eigenvalues, we can restore the dependence of the energy levels in g by multiplying the results by a factor of $g^{2/3}$. The restoration of the corresponding wave functions (or trial functions), can also be obtained by rescaling $x \rightarrow g^{1/3}x$ in the results.

To go on with the variational method, we construct the expectation value of the energy with these trial functions:

$$E = \frac{\langle \phi | H_{\pm} | \phi \rangle}{\langle \phi | \phi \rangle} = \frac{\sum_{k=1}^m \sum_{l=1}^m \alpha_k \alpha_l \langle f_k | H_{\pm} | f_l \rangle}{\sum_{k=1}^m \sum_{l=1}^m \alpha_k \alpha_l \langle f_k | f_l \rangle} \quad (31)$$

and minimize E with respect to the parameters α_l . This condition gives the system of linear equations:

$$\sum_{l=1}^m ((H_{\pm})_{kl} - ES_{kl})\alpha_l = 0, \quad (32)$$

where we used the notation $H_{kl} = \langle f_k | H | f_l \rangle$ and $S_{kl} = \langle f_k | f_l \rangle$. The values of E that minimize the above system of equations are the eigenvalues of the matrix:

$$M_{kl} = (ES_{kl} - (H_{\pm})_{kl}) \quad (33)$$

and are obtained by solving the equation $\det M = 0$. The wave functions corresponding to each of these eigenvalues are got by substituting the value of E in the linear system above and solving for the parameters α_k . The matrix elements that we need to construct M_{kl} are:

$$S_{kl} = \langle f_k | f_l \rangle = \int_{-\infty}^{+\infty} dx e^{-\frac{2}{3}|x|^3} x^{k+l-2} \quad (34)$$

$$(H_{\pm})_{kl} = \langle f_k | H_{\pm} | f_l \rangle = \int_{-\infty}^{+\infty} dx e^{-\frac{2}{3}|x|^3} [-(l-1)(l-2)x^{k+l-4} + 2(l \pm 1)\varepsilon(x)x^{k+l-1}] . \quad (35)$$

For $(k+l)$ odd, the integrands in (34) and (35) are odd functions and $S_{kl} = (H_{\pm})_{kl} = 0$. Otherwise, for $(k+l)$ even, we find:

$$S_{kl} = \left(\frac{3}{2}\right)^{\frac{k+l-4}{3}} \Gamma\left(\frac{k+l-1}{3}\right) \quad (36)$$

$$(H_{\pm})_{kl} = -2 \left(\frac{3}{2}\right)^{\frac{k+l-3}{3}} \left[\frac{(l-1)(l-2) - (l \pm 1)(k+l-3)}{(k+l-3)} \right] \Gamma\left(\frac{k+l}{3}\right) . \quad (37)$$

With these results, the matrix M (33) gets the form:

$$M_{\pm} = \begin{pmatrix} (M_{\pm})_{11} & 0 & (M_{\pm})_{13} & 0 & \dots & (M_{\pm})_{1m} \\ 0 & (M_{\pm})_{22} & 0 & (M_{\pm})_{24} & \dots & (M_{\pm})_{2m} \\ (M_{\pm})_{31} & 0 & (M_{\pm})_{33} & 0 & \dots & (M_{\pm})_{3m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (M_{\pm})_{m1} & (M_{\pm})_{m2} & (M_{\pm})_{m3} & (M_{\pm})_{m4} & \dots & (M_{\pm})_{mm} \end{pmatrix}. \quad (38)$$

In this matrix, all elements in positions (k, l) , such that $(k + l)$ is odd are null, while those with $(k + l)$ even are given by (33) with S_{kl} and H_{kl} respectively given by (36) and (37). To find the energy values we must solve the equation: $\det M = 0$.

Tables I and II show some results found for different number (m) of parameters and for $g = 1$. For different values of g , the values in the Table must be multiplied by a factor of $g^{2/3}$, as observed above.

TABLE I: Energy values associated with H_- calculated for different numbers of variational parameters.

m	$E_0^-^a$	E_1^-	E_2^-	E_3^-	E_4^-	E_5^-	E_6^-	E_7^-
1	0.00000							
2	0.00000	2.04441						
3	0.00000	2.04441	5.76541					
4	0.00000	1.97852	5.76541	10.00191				
5	0.00000	1.97852	5.54135	10.00191	14.94174			
6	0.00000	1.97115	5.54135	9.49446	14.94174	20.37028		
7	0.00000	1.97115	5.51302	9.49446	14.06558	20.37028	26.29953	
8	0.00000	1.96991	5.51302	9.41370	14.06558	19.02962	26.29953	32.64399
9	0.00000	1.96991	5.50842	9.41370	13.90148	19.02962	24.43194	32.64399
10	0.00000	1.96963	5.50842	9.39868	13.90148	18.73498	24.43194	30.18755

^aFor this level, the variational method provides the exact solution.

TABLE II: Energy values associated with H_+ calculated for different numbers of variational parameters.

m	E_0^+	E_1^+	E_2^+	E_3^+	E_4^+	E_5^+	E_6^+
1	2.31447						
2	2.31447	6.13324					
3	2.04493	6.13324	10.54940				
4	2.04493	5.63655	10.54940	15.63469			
5	1.99066	5.63655	9.66470	15.63469	21.21933		
6	1.99066	5.53888	9.66470	14.30956	21.21933	27.28556	
7	1.97666	5.53888	9.46567	14.30956	19.36916	27.28556	33.76558
8	1.97666	5.51611	9.46567	13.98107	19.36916	24.86727	33.76558
9	1.97235	5.51611	9.41524	13.98107	18.85787	24.86727	30.72924
10	1.97235	5.51007	9.41524	13.89369	18.85787	24.13659	30.72924

The results in Table I and II reflect the manifestation of SUSY in the system, at least with respect to the equality between the energy levels E_n^- and E_{n-1}^+ , $n > 0$, of H_- and H_+ . As expected, the ground state energy of H_- is zero and it is not equal to any energy of H_+ . Moreover, for $n > 0$, increasing the number of variational parameters, we find, mainly for the first levels, energies E_n^- more and more closer to E_{n-1}^+ .

Therefore, the better the trial we make, the closer we are to satisfy the equality between energy levels. Moreover, because the one parameter trial function for the ground state of H_- has the same form of the exact (analytical) solution, the value $E_0^- = 0$ found is exact and the condition of having a zero energy ground state is naturally satisfied.

Figure 2 shows the first energy levels of H_- and H_+ . We must remember that the values found are better for increasing number of variational parameters and for the lowest levels. Thus, for instance, we are supposed to find for the level $n = 4$ a worse approximation than for the level $n = 1$.

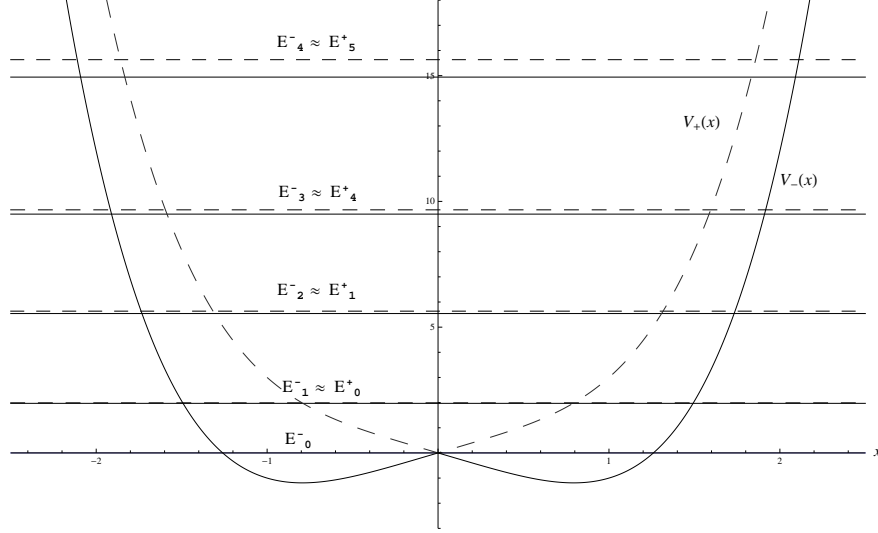


FIG. 2: Scheme for the 5 first levels of H_- (and 4 first levels of H_+) using 6 variational parameters.

The graphics in Fig. 3 show the approximations for the first levels eigenfunctions of H_- and H_+ , respectively. Those approximations were found using 6 variational parameters.

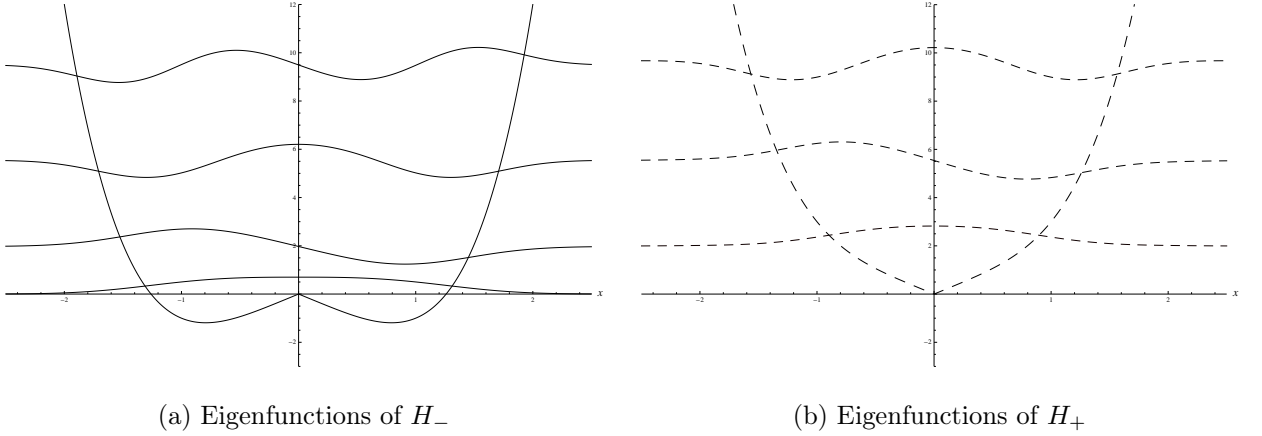


FIG. 3: Eigenfunctions of the first levels of H_- and H_+ for 6 variational parameters.

As expected, we see that the eigenfunctions found have well defined parity, interchanging even and odd solutions with even solutions for the ground states.

B. Looking for Approximate Solutions by a Logarithmic Perturbation Theory

We now apply a variant of the logarithmic perturbation theory (LPT) to our problem. LPT is explained with more details, for example, in [3], [7], [8], [9] or [11].

Starting from the known solution ψ_0^- of V_- we can perturbatively obtain the ground state of V_+ , or for example, of the anharmonic potential $V(x) = x^4$. We start by writting:

$$V(x; \delta) = V_0(x) + \delta V_1(x) , \quad (39)$$

where:

$$V_0(x) = V_-(x) = x^4 - 2|x| \quad (40)$$

$$V_1(x) = 4|x| . \quad (41)$$

Observe that $V(x; \delta = 1) = V_+$ and that $V(x; \delta = 1/2) = x^4$. As we only know the ground state of V_- we can not go beyond the first order in the Rayleigh-Schrödinger perturbation theory. To bypass this difficult we will use the so called logarithmic perturbation theory where only the knowledge ψ_0^- is required to calculate the ground state energy level of $V(x; \delta)$ to any order in δ (at least numerically). For that aim we consider the perturbed Schrödinger equation:

$$-\Psi'' + (V_0 + \delta V_1)\Psi = E\Psi \quad (42)$$

and write the expansions:

$$E = E_0 + \delta E_1 + \delta^2 E_2 + \dots \quad (43)$$

$$\Psi = \exp(S_0 + \delta S_1 + \delta^2 S_2 + \dots) , \quad (44)$$

where S_1, S_2 , etc. are functions and E_1, E_2 , etc. are numbers to be determined. By substituting these expressions in the Schrödinger equation above and equating the terms of same powers in δ we get the set of equations:

$$S_0'' + S_0'^2 = -E_0 + V_2 \quad (45)$$

$$S_1'' + 2S_0'S_1' = -E_1 + V_1 \quad (46)$$

$$S_2'' + 2S_0'S_2' + S_1'^2 = -E_2 \quad (47)$$

$$S_3'' + 2S_0'S_3' + 2S_1'S_2' = -E_3 \quad (48)$$

\vdots

Starting with $E_0 = 0$ and $S_0 = -|x|^3/3$, (that is, $\Psi_0(x) = \psi_0^- = \mathcal{N}e^{-|x|^3/3}$), these equations can be recursively solved to get E_k and S_k to the desired order in δ .

Eq. (46) can be rewritten as:

$$(S'_1 \exp(2S_0))' = (V_1 - E_1) \exp(2S_0) . \quad (49)$$

By substituting $S_0 = -|x|^3/3$ and $V_1 = 4|x|$ in this equation, integrating both sides in the interval $x = (-\infty, +\infty)$ and observing that the integrand of the left side goes exponentially to zero at both ends of the integration range, we get for E_1 the result:

$$\begin{aligned} E_1 &= \frac{\langle \psi_0 | V_1(x) | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = \frac{\int_{-\infty}^{+\infty} dx e^{-\frac{2}{3}|x|^3} 4|x|}{\int_{-\infty}^{+\infty} dx e^{-\frac{2}{3}|x|^3}} \\ &= 4 \left(\frac{3}{2} \right)^{1/3} \frac{\Gamma(2/3)}{\Gamma(1/3)} = 2.31447 . \end{aligned} \quad (50)$$

Inserting this result for E_1 back into the same equation and integrating now in the interval $y = (0, x)$ we get:

$$\begin{aligned} S'_1(x) &= |\psi_0(x)|^{-2} \int_0^x dy |\psi_0(y)|^2 [E_1 - V_1(y)] \\ &= e^{\frac{2}{3}|x|^3} \int_0^x dy e^{-\frac{2}{3}|y|^3} \left[4 \left(\frac{3}{2} \right)^{1/3} \frac{\Gamma(2/3)}{\Gamma(1/3)} - 4|y| \right] \\ &= -2 \left(\frac{2}{3} \right)^{1/3} e^{\frac{2}{3}|x|^3} \left[\frac{\Gamma(2/3)}{\Gamma(1/3)} \Gamma(1/3, 2x^3/3) - \Gamma(2/3, 2x^3/3) \right] , \end{aligned} \quad (51)$$

where $\Gamma(\alpha, x) \equiv \int_x^\infty dt e^{-t} t^{\alpha-1}$ are the upper incomplete gamma functions[20].

The second order equation (47) can also be written in the form:

$$(S'_2 \exp(2S_0))' = (-S_1'^2 - E_2) \exp(2S_0) . \quad (52)$$

Integrating this equation in the interval $x = (-\infty, +\infty)$, and observing that the integrand of the left side goes to zero at both ends of the integration range, we get E_2 as an integral over S'_1 :

$$E_2 = -\frac{\langle \psi_0 | S_1'^2(x) | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = -\frac{3}{\Gamma(1/3)} \left(\frac{2}{3} \right)^{1/3} \int_0^\infty dx e^{-\frac{2}{3}|x|^3} S_1'(x)^2 . \quad (53)$$

Substituting (51) in (53), we find:

$$E_2 = -\frac{4}{\Gamma(1/3)} \left(\frac{2}{3} \right)^{2/3} \left\{ \frac{\Gamma(2/3)^2}{\Gamma(1/3)^2} I_{2/3} \left(\frac{1}{3}, \frac{1}{3} \right) + I_{2/3} \left(\frac{2}{3}, \frac{2}{3} \right) - 2 \frac{\Gamma(2/3)}{\Gamma(1/3)} I_{2/3} \left(\frac{1}{3}, \frac{2}{3} \right) \right\} , \quad (54)$$

where:

$$I_\alpha(x, y) = \int_0^\infty dt e^{t t^{-\alpha}} \Gamma(x, t) \Gamma(y, t), \quad x > 0, \quad y > 0, \quad 0 < \alpha < 1. \quad (55)$$

Evaluating the integrals, the expression (54) gives $E_2 = -0.43817$.

Sumarizing: up to the second order, the ground state energy of $V(x; \delta)$ is given by:

$$E(\delta) = E_0 + \delta E_1 + \delta^2 E_2, \quad (56)$$

with $E_0 = 0$, $E_1 = 2.31447$ and $E_2 = -0.43817$.

For $\delta = 1$, we get for the ground state energy of V_+ , the result: $E_0^+ = 1.87630$.

For $\delta = 1/2$, we find the result $E_0^{x^4} = 1.04769$ for the ground state energy of the the quartic anharmonic potential $V(x) = x^4$. This result can be compared with the exact one given in [3], noting that our “coupling” constant g is related to their constant \tilde{g} by: $g^{2/3} = (\frac{1}{4})^{1/3} \tilde{g}^{1/3}$. Thus, multiplying our result by $(\frac{1}{4})^{1/3}$, we find: $\tilde{E}_0^{x^4} = 0.66000$, differing of [3] only by about 1.2%.

On the other hand, comparing the value of E_0^+ found here with the most accurate result of the variational method (see Table II), we see that they differ by about 4.9%, what does not seem very good. But, following the suggestion of [8] or [21], and substituting the expression (56) by the corresponding $[1, 1]$ Padé approximant in δ , we find:

$$E(\delta) = \frac{E_0 E_1 + (E_1^2 - E_0 E_2) \delta}{E_1 - E_2 \delta}, \quad (57)$$

which results (for $\delta = 1$) in $E_0^+ = 1.94605$. This result now differs from the result of Table II only by 1.3%. Doing the same for $\delta = 1/2$ (and then multiplying by $(\frac{1}{4})^{1/3}$), we find $\tilde{E}_0^{x^4} = 0.66597$, differing from the result of Cooper et al [3] by only 0.03%. A pretty good result.

IV. CONCLUSIONS

In this paper we studied the class of superpotentials $W(x) = \varepsilon(x)x^{2n}$ in SUSY QM. After revisiting the case $n = 0$ we went on studying in details the case $W(x) = \varepsilon(x)x^2$. As a result we got the exact solution for the ground state of the potential $V_-(x) = x^4 - 2|x|$, showed that exact solutions do not exist for the excited states and studied these states by a variational method. Finally, starting from the known ground state of $V(x) = x^4 - 2|x|$,

we obtained the ground states for the potentials $V(x) = x^4$ and $V(x) = x^4 + 2|x|$, by using logarithmic perturbation theory. Comparison with other known results in the literature and in the paper are given.

Some other approaches and improvements can be used to study this class of superpotentials. In a forthcoming paper we analyze the solutions for the ground states of $V_{\pm}(x) = x^4 \pm 2|x|$ by starting with the solutions of $V_{\pm} = x^2 \pm C$ and using LPT and the δ expansion of Bender [7] and Cooper [8].

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